

Independence polynomials of well-covered graphs: generic counterexamples for the unimodality conjecture

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Abstract

A graph G is *well-covered* if all its maximal stable sets have the same size, denoted by $\alpha(G)$ (M. D. Plummer, 1970). If s_k denotes the number of stable sets of cardinality k in graph G , and $\alpha(G)$ is the size of a maximum stable set, then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the *independence polynomial* of G (I. Gutman and F. Harary, 1983). J. I. Brown, K. Dilcher and R. J. Nowakowski (2000) conjectured that $I(G; x)$ is unimodal (i.e., there is some $j \in \{0, 1, \dots, \alpha(G)\}$ such that $s_0 \leq \dots \leq s_{j-1} \leq s_j \geq s_{j+1} \geq \dots \geq s_{\alpha(G)}$) for any well-covered graph G . T. S. Michael and W. N. Traves (2002) proved that this assertion is true for $\alpha(G) \leq 3$, while for $\alpha(G) \in \{4, 5, 6, 7\}$ they provided counterexamples.

In this paper we show that for any integer $\alpha \geq 8$, there exists a connected well-covered graph G with $\alpha = \alpha(G)$, whose independence polynomial is not unimodal. In addition, we present a number of sufficient conditions for a graph G with $\alpha(G) \leq 6$ to have the unimodal independence polynomial.

key words: *stable set, independence polynomial, unimodal sequence, well-covered graph.*

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. $K_n, P_n, K_{n_1, n_2, \dots, n_p}$ denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 3$ vertices, and the complete p -partite graph on $n_1 + n_2 + \dots + n_p$ vertices, where $n_i \geq 1, 1 \leq i \leq p$.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \sqcup G_2$ having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. In particular, $\sqcup n G$ denotes the disjoint union of $n > 1$ copies of the graph G . The *Zykov sum* ([25], [26]) of two disjoint graphs G_1, G_2 is the graph $G_1 + G_2$ that has $V(G_1) \cup V(G_2)$ as a vertex set and $E(G_1) \cup E(G_2) \cup \{v_1 v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$ as an edge set.

A *stable* set in G is a set of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ of G is the maximum size of a stable set in G . By $\omega(G)$ we mean $\alpha(\overline{G})$, where \overline{G} is the complement of G .

A graph G is called *well-covered* if all its maximal stable sets are of the same cardinality, (Plummer, [22]). If, in addition, G has no isolated vertices and its order equals $2\alpha(G)$, then G is *very well-covered* (Favaron, [6]). For instance, the graph G^* , obtained from G by appending a single pendant edge to each vertex of G ([5], [24]), is well-covered (see, for example, [15]), and $\alpha(G^*) = n$. Moreover, G^* is very well-covered, since it is well-covered, it has no isolated vertices, and its order equals $2\alpha(G^*)$. The following result shows that, under certain conditions, any well-covered graph equals G^* for some graph G .

Theorem 1.1 [7] *Let H be a connected graph of girth ≥ 6 , which is isomorphic to neither C_7 nor K_1 . Then H is well-covered if and only if its pendant edges form a perfect matching.*

In other words, Theorem 1.1 shows that apart from K_1 and C_7 , connected well-covered graphs of girth ≥ 6 are very well-covered. For example, a tree $T \neq K_1$ could be only very well-covered, and this is the case if and only if $T = G^*$ for some tree G (see also Ravindra, [23]).

Let s_k be the number of stable sets in G of cardinality $k \in \{0, 1, \dots, \alpha(G)\}$. The polynomial $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is called the *independence polynomial* of G (Gutman and Harary, [10]). It is easy to deduce that

$$\begin{aligned} I(G_1 \sqcup G_2; x) &= I(G_1; x) \cdot I(G_2; x), \\ I(G_1 + G_2; x) &= I(G_1; x) + I(G_2; x) - 1 \end{aligned}$$

(see also [10], [2], [13]).

A finite sequence of real numbers $(a_0, a_1, a_2, \dots, a_n)$ is said to be:

- *unimodal* if there is some $k \in \{0, 1, \dots, n\}$, called the *mode* of the sequence, such that

$$a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n,$$

- *log-concave* if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$.

It is known that any log-concave sequence of positive numbers is also unimodal, while the converse is not generally true.

A polynomial $P = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ is called *unimodal (log-concave)* if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, the independence polynomial $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$ is log-concave, while

$$I(K_{25} + (K_3 \sqcup K_4 \sqcup K_5 \sqcup K_5); x) = 1 + 42x + \mathbf{107}x^2 + 295x^3 + 300x^4$$

is unimodal, but it is not log-concave, because $107^2 - 42 \cdot 295 = -941$.

Hamidoune [12] proved that the independence polynomial of a *claw-free* graph (i.e., a graph having no induced subgraph isomorphic to $K_{1,3}$) is log-concave, and hence, unimodal. However, there are graphs whose independence polynomials are not unimodal, e.g., $I(K_{70} + (\sqcup 4K_3); x) = 1 + \mathbf{82}x + 54x^2 + \mathbf{108}x^3 + 81x^4$ (for other examples, see [1]). Nevertheless, in [1] it is stated the following (still open) *unimodality conjecture* for trees.

Conjecture 1.2 *The independence polynomial of any tree is unimodal.*

In [17] and [18], the unimodality of independence polynomials of a number of well-covered trees (e.g., $P_n^*, K_{1,n}^*$) is validated, using the result, mentioned above, on claw-free graphs due to Hamidoune, or directly, by identifying the location of the mode. These findings seem promising for proving Conjecture 1.2 in the case of very well-covered trees, since a tree T is well-covered if and only if either T is a *well-covered spider* (i.e., $T \in \{K_1, K_1^*, K_{1,n}^* : n \geq 1\}$), or T is obtained from a well-covered tree H_1 and a well-covered spider H_2 , by adding an edge joining two non-pendant vertices belonging to H_1, H_2 , respectively (see [16]). For instance, the trees presented in Figure 1 are well-covered as follows: T_2 is a well-covered spider, while T_1 is an edge-join of two well-covered spiders, namely, $K_{1,2}^*$ and $K_{1,1}^*$.

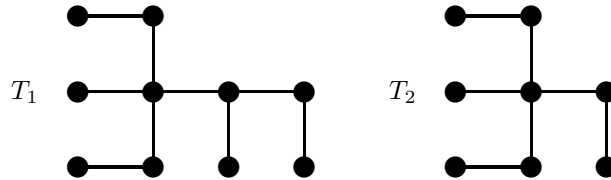


Figure 1: Two well-covered trees.

In [3] it was conjectured that the independence polynomial of any well-covered graph is unimodal. Michael and Traves [21] proved that this assertion is true for $\alpha(G) \in \{1, 2, 3\}$, but it is false for $\alpha(G) \in \{4, 5, 6, 7\}$. Nevertheless, the conjecture of Brown *et al.* is still open for very well-covered graphs.

In [20] it was shown that for any $\alpha \geq 1$, there is a connected very well-covered graph G with $\alpha(G) = \alpha$, whose independence polynomial is unimodal.

In this paper we prove that for any integer number $\alpha \geq 8$, there exists a connected well-covered graph G with $\alpha(G) = \alpha$, whose $I(G; x)$ is not unimodal. We also give a simple proof for the unimodality of the independence polynomial of a well-covered graph G with $\alpha(G) \leq 3$, while for $\alpha(G) \in \{4, 5, 6\}$ a number of sufficient conditions ensuring the unimodality of $I(G; x)$ are presented.

2 The small stability number as a reason for well-covered graphs to have unimodal independence polynomials

Alavi *et al.* [1] showed that for any permutation σ of $\{1, 2, \dots, \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that $s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(\alpha)}$.

Lemma 2.1 *If a graph G satisfies $\omega(G) \leq \alpha = \alpha(G)$, then $s_\alpha \leq s_{\alpha-1}$.*

Proof. Let $H = (\mathcal{A}, \mathcal{B}, \mathcal{W})$ be the bipartite graph defined as follows: $X \in \mathcal{A} \Leftrightarrow X$ is a stable set in G of size $\alpha(G) - 1$, then $Y \in \mathcal{B} \Leftrightarrow Y$ is a stable set in G of size $\alpha(G)$, and $XY \in \mathcal{W} \Leftrightarrow X \subset Y$ in G . Since any $Y \in \mathcal{B}$ has exactly $\alpha(G)$ subsets of size $\alpha(G) - 1$, it follows that $|\mathcal{W}| = \alpha(G) \cdot s_\alpha$. On the other hand, if $X \in \mathcal{A}$ and $X \cup \{y_1\}, X \cup \{y_2\} \in \mathcal{B}$, it implies $y_1 y_2 \in E(G)$, because X is stable and $|X \cup \{y_1, y_2\}| > \alpha(G)$. Hence, any $X \in \mathcal{A}$ has at most $\omega(G)$ neighbors. Consequently, $|\mathcal{W}| = \alpha(G) \cdot s_\alpha \leq \omega(G) \cdot s_{\alpha-1}$, and this leads to $s_\alpha \leq s_{\alpha-1}$, since $\alpha(G) \geq \omega(G)$. ■

The converse of Lemma 2.1 is not true, e.g., $\alpha(K_4 - e) = 2 < 3 = \omega(K_4 - e)$ and $I(K_4 - e; x) = 1 + 4x + x^2$, where by $K_4 - e$ we mean the graph obtained from K_4 by deleting one of its edges.

Proposition 2.2 [21], [19] *If G is a well-covered graph having $\alpha(G) = \alpha$, then $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$.*

Corollary 2.3 *If G is a well-covered graph and $\omega(G) \leq \alpha(G) = 3$, then $I(G; x)$ is log-concave.*

Proof. Let $I(G; x) = s_0 + s_1 x + s_2 x^2 + s_3 x^3$. By Proposition 2.2 and Lemma 2.1, we get $s_0 \leq s_1 \leq s_2 \geq s_3$, which implies that $s_2^2 \geq s_1 s_3$. To complete the proof, let us notice that $s_1^2 = |V(G)|^2 \geq |E(\overline{G})| = s_2 = s_0 s_2$. ■

The roots of the independence polynomials of well-covered graphs are investigated in a number of papers, as [3], [4], [8], [9], [11], [19]. Brown *et al.* showed, by a nice argument, that:

Lemma 2.4 [3] *If a graph G has $\alpha(G) = 2$, then $I(G; x)$ has real roots.*

The assertion fails for graphs with stability number greater than 2, e.g., $I(K_{1,3}; x)$. Notice that the independence polynomials of the trees from Figure 1, are respectively

$$\begin{aligned} I(T_1; x) &= 1 + 10x + 36x^2 + \mathbf{60}x^3 + 47x^4 + 14x^5, \\ I(T_2; x) &= 1 + 8x + 21x^2 + \mathbf{23}x^3 + 9x^4, \end{aligned}$$

while only for the first is true that all its roots are real. Let us observe that T_1, T_2 are well-covered and their polynomials are unimodal. Hence, Newton's theorem (stating that if a polynomial with positive coefficients has only real roots, then its coefficients

form a log-concave sequence) is not useful in solving Conjecture 1.2, even for the particular case of very well-covered trees.

Let us mention that there are connected graphs, with stability number equal to 3, whose independence polynomials are:

- not unimodal, e.g.,

$$I(K_{24} + (K_4 \sqcup K_3 \sqcup K_3)); x) = 1 + \mathbf{34}x + 33x^2 + \mathbf{36}x^3;$$

- unimodal, but not log-concave, e.g.,

$$I(K_{95} + (\sqcup 3K_7)); x) = 1 + 116x + \mathbf{147}x^2 + 343x^3;$$

- unimodal, but not log-concave, while the graphs are also well-covered, e.g.,

$$I((\sqcup 3K_{10}) + \underbrace{K_3, 3, \dots, 3}_{120}; x) = 1 + 390x + \mathbf{660}x^2 + 1120x^3.$$

There are also well-covered connected graphs with stability number equal to 4, whose independence polynomials are:

- not unimodal, e.g.,

$$I((\sqcup 4K_{10}) + \underbrace{K_4, 4, \dots, 4}_{1800}; x) = 1 + 7240x + \mathbf{11400}x^2 + 11200x^3 + \mathbf{11800}x^4;$$

- unimodal, but not log-concave, e.g.,

$$I((\sqcup 4K_{10}) + \underbrace{K_4, 4, \dots, 4}_{25}; x) = 1 + 140x + \mathbf{750}x^2 + 4100x^3 + 10025x^4;$$

- log-concave, e.g.,

$$I((\sqcup 4K_{10}) + \underbrace{K_4, 4, \dots, 4}_{10}; x) = 1 + 80x + 660x^2 + 4040x^3 + 10010x^4.$$

Let us observe that the product of two unimodal independence polynomials is not always unimodal, e.g., $I(K_{100} + \sqcup 3K_7; x) = 1 + 121x + 147x^2 + \mathbf{343}x^3$ and $I(K_{90} + \sqcup 3K_7; x) = 1 + 111x + 147x^2 + \mathbf{343}x^3$, while their product is not unimodal:

$$1 + 232x + 13725x^2 + 34790x^3 + \mathbf{101185}x^4 + 100842x^5 + \mathbf{117649}x^6.$$

Theorem 2.5 [14] *The product of a log-concave polynomial by a unimodal polynomial is unimodal, while the product of two log-concave polynomials is log-concave.*

Theorem 2.5 is best possible for independence polynomials, since the product of a log-concave independence polynomial and a unimodal independence polynomial is not always log-concave. For instance, $I(K_{40} + \sqcup 3K_7; x) = 1 + 61x + 147x^2 + \mathbf{343}x^3$ is log-concave, $I(K_{110} + \sqcup 3K_7; x) = 1 + 131x + 147x^2 + \mathbf{343}x^3$ is unimodal, while their product

$$1 + 192x + 8285x^2 + 28910x^3 + 87465x^4 + \mathbf{100842}x^5 + 117649x^6$$

is not log-concave.

Further we summarize some facts on graphs with small stability numbers.

Proposition 2.6 *The following is a list of sufficient conditions ensuring that the independence polynomial of a graph G is unimodal:*

- (i) *any connected component H of G has $\alpha(H) \leq 2$;*
- (ii) *$\alpha(G) = 3$ and G is well-covered;*
- (iii) *$\alpha(G) = 4$, G is disconnected and well-covered;*
- (iv) *$\alpha(G) = 5$, $G = H_1 \sqcup H_2$, $\alpha(H_1) = 2$ and H_2 is well-covered;*
- (v) *$\omega(G) \leq \alpha(G) \leq 5$ and G is well-covered;*
- (vi) *$\alpha(G) = 6$, G is disconnected and any component H of G with $\alpha(H) \in \{3, 4, 5\}$ is well-covered and satisfies $\omega(H) \leq \alpha(H)$.*

Proof. (i) If H_1, H_2, \dots, H_k are the components of G and $\alpha(H_i) \leq 2, 1 \leq i \leq k$, then $I(G; x)$ is unimodal, by Newton's Theorem, because $I(G; x) = I(H_1; x) \cdot \dots \cdot I(H_k; x)$ and, consequently, by Lemma 2.4, all its roots are real.

(ii) If G is disconnected, then $I(G; x)$ is unimodal, by part (i). Assume that G is connected, and let $I(G; x) = 1 + nx + s_2x^2 + s_3x^3$, where n is the order of G . Any vertex $v \in V(G)$ is contained in some maximum stable set of G , since G is well-covered. Hence, v has at least two neighbors in the complement \overline{G} of G , which ensures that $n \leq |E(\overline{G})| = s_2$. Consequently, $I(G; x)$ is unimodal, with the mode 2 or 3, depending on $\max\{s_2, s_3\}$, respectively. Let us mention that there are connected well-covered graphs with stability number equal to 3, whose independence polynomial has non-real roots, e.g., $I(K_{3,3,3}; x) = 1 + 9x + 9x^2 + 3x^3$ has non-real roots.

(iii) If G is disconnected and at least one of its components is a complete graph, then $G = K_p \sqcup H$ and $I(G; x) = I(K_p; x) \cdot I(H; x) = (1 + px) \cdot (1 + s_1x + s_2x^2 + s_3x^3)$ is unimodal, by Theorem 2.5. If none of its components is a complete graph, then G has only two components, say H_1 and H_2 , and $\alpha(H_1) = \alpha(H_2) = 2$. Hence, by Lemma 2.4, $I(H_1; x), I(H_2; x)$ have only real roots. Therefore, $I(G; x) = I(H_1; x) \cdot I(H_2; x)$ is unimodal, by Newton's Theorem.

(iv) According to Lemma 2.4 and Newton's Theorem, $I(H_1; x)$ is log-concave. Since $G = H_1 \sqcup H_2$, it follows that $I(G; x) = I(H_1; x) \cdot I(H_2; x)$. Hence, using part (ii) and Theorem 2.5, we infer that $I(G; x)$ is unimodal.

(v) Taking into account the parts (i), (ii), we may assume that $\alpha(G) \in \{4, 5\}$.

Suppose that $\alpha(G) = 4$. Then, $I(G; x) = s_0 + s_1x + s_2x^2 + s_3x^3 + s_4x^4$, and, according to Proposition 2.2, we obtain that $s_0 \leq s_1 \leq s_2$, since G is well-covered, while by Lemma 2.1, it follows that $s_3 \geq s_4$, because $\omega(G) \leq \alpha(G)$. Therefore, $I(G; x)$ is unimodal, with the mode 2 or 3, depending on $\max\{s_2, s_3\}$. Now, for $\alpha(G) = 5$,

$I(G; x) = s_0 + s_1x + s_2x^2 + s_3x^3 + s_4x^4 + s_5x^5$ and Proposition 2.2 implies that $s_0 \leq s_1 \leq s_2 \leq s_3$, while Lemma 2.1 assures that $s_4 \geq s_5$, since $\alpha(G) \geq \omega(G)$. Consequently, $I(G; x)$ is unimodal, with the mode 3 or 4, depending on $\max\{s_3, s_4\}$.

(vi) If G has a component H with $\alpha(H) \in \{4, 5\}$, this is unique, and $\alpha(G - H) \leq 2$. Consequently, by parts (i), (v) and Theorem 2.5, $I(G; x) = I(H; x) \cdot I(G - H; x)$ is unimodal. If G has two components H_1, H_2 with $\alpha(H_1) = \alpha(H_2) = 3$, then Corollary 2.3 and Theorem 2.5 assure that $I(G; x) = I(H_1; x) \cdot I(H_2; x)$ is unimodal. The other cases follow easily, by applying parts (i), (iii) and Theorem 2.5. ■

3 A family of well-covered graphs having non-unimodal independence polynomials

The independence polynomial of $H_n = (\sqcup 4K_{10}) + \underbrace{K_{4, 4, \dots, 4}}_n, n \geq 1$ is

$$\begin{aligned} I(H_n; x) &= n \cdot (1 + x)^4 + (1 + 10x)^4 - n \\ &= 1 + (40 + 4n)x + (600 + 6n)x^2 + (4000 + 4n)x^3 + (10000 + n)x^4. \end{aligned}$$

Let us notice that $\alpha(H_n) = 4$ and H_n is well-covered. Since $40 + 4n < 600 + 6n$ is true for any $n \geq 1$, it follows that $I(H_n; x)$ is not unimodal whenever

$$4000 + 4n < \min\{600 + 6n, 10000 + n\},$$

which leads to $1700 < n < 2000$, where the case $n = 1701$ is due to Michael and Traves, [21]. Moreover, $I(H_n; x)$ is not log-concave only for $23 < n < 2453$.

Lemma 3.1 *For any integer $k \geq 0$, the following polynomial is not unimodal.*

$$\sum_{i=0}^{k+4} s_i x^i = (1 + 6844 \cdot x + 10806 \cdot x^2 + 10804 \cdot x^3 + 11701 \cdot x^4) \cdot (1 + 1000 \cdot k \cdot x)^k.$$

Proof. We show that $s_{k+2} > s_{k+3} < s_{k+4}$. Since the result is evident for $k = 0$, we may assume that $k \geq 1$.

Let us notice that:

$$\begin{aligned} s_{k+4} &= 11701 \cdot 10^{3k} \cdot k^k, \\ s_{k+3} &= 10804 \cdot 10^{3k} \cdot k^k + 11701 \cdot 10^{3(k-1)} \cdot k^k = 10^{3(k-1)} \cdot k^k \cdot 10815701, \\ s_{k+2} &= 10806 \cdot 10^{3k} \cdot k^k + 10804 \cdot 10^{3(k-1)} \cdot k^k + \\ &\quad + 11701 \cdot 10^{3(k-2)} \cdot k^{k-1} \cdot (k-1) \cdot 0.5 \\ &= 10^{3(k-2)} \cdot k^{k-1} \cdot (2 \cdot 16336 \cdot 19701 \cdot k - 11701) \cdot 0.5. \end{aligned}$$

Firstly, we have

$$\begin{aligned} s_{k+4} - s_{k+3} &= 11701 \cdot 10^{3k} \cdot k^k - 10^{3(k-1)} \cdot k^k \cdot 10815701 \\ &= 10^{3(k-1)} \cdot k^k \cdot 885299 > 0. \end{aligned}$$

Secondly, we obtain

$$\begin{aligned}
s_{k+2} - s_{k+3} &= 10^{3(k-2)} \cdot k^{k-1} \cdot (2\,16336\,19701 \cdot k - 11701) \cdot 0.5 \\
&\quad - 10^{3(k-1)} \cdot k^k \cdot 10815701 \\
&= 10^{3(k-2)} \cdot k^{k-1} \cdot (2217701 \cdot k - 11701) \cdot 0.5 > 0,
\end{aligned}$$

which completes the proof. ■

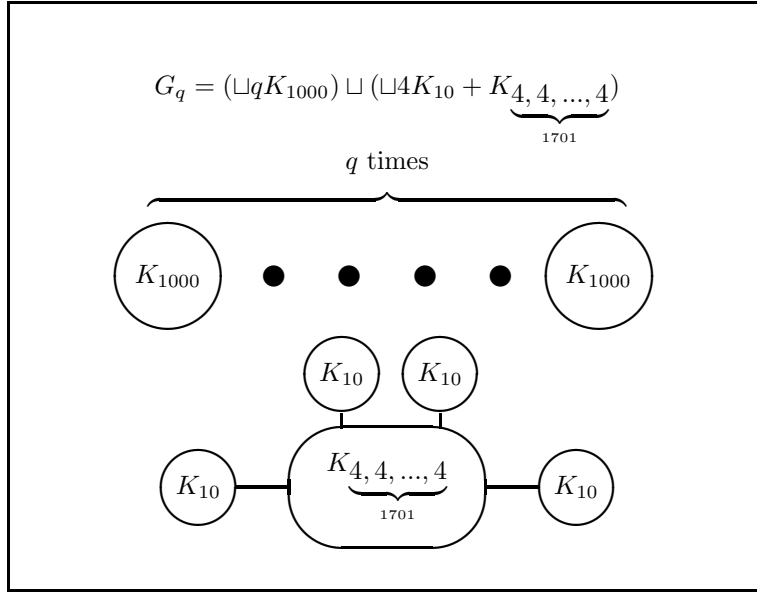


Figure 2: Well-covered graphs with non-unimodal independence polynomials.

Theorem 3.2 *For any integer $k \geq 4$, there is a well-covered graph G with $\alpha(G) = k$, whose independence polynomial is not unimodal.*

Proof. Let $q = k - 4$ and G_q be the graph depicted in Figure 2, and formally defined as follows:

$$G_q = (\sqcup q K_{1000}) \sqcup (\sqcup 4 K_{10} + K_{4,4,\dots,4}).$$

1701

It is easy to see that G_q is a disconnected well-covered graph, $\alpha(G_q) = k$, and its independence polynomial is not unimodal, because $I(G_q; x)$ is identical to the non-unimodal polynomial from Lemma 3.1.

Moreover, the graph $G_q + G_q$ is well-covered, connected, $\alpha(G_q + G_q) = k$, and its independence polynomial is not unimodal, since $I(G_q + G_q; x) = 2 \cdot I(G_q; x) - 1$. ■

4 Conclusions

In this paper we demonstrated that for every integer $k \geq 8$ there exists a (dis)connected well-covered graph G with $\alpha(G) = k$, whose independence polynomial is not unimodal. It is worth mentioning that all these graphs are not very well-covered. In other words, the unimodality conjecture remains open for the case of very well-covered graphs.

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